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Generalization of the Dirac Equation Admitting Isospin and Color Symmetries

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One possible generalization of the Dirac "square root" procedure $\partial_\mu \partial^\mu = D_d^\kappa D_d$ is presented, based on the explicit introduction of chiral symmetry, which generates a set $\{d\}$ of symmetry-constrained Dirac fields $D_d \psi_d = 0$ admitting isospin and color. A self-consistent discussion is given of the basic geometrical construction, the field equations, and their relationship to chiral symmetry, isospin, and color, and of the construction of the Lagrangian, including the interaction gauge fields. The correspondence of the theory with the standard $SU_c(3) \times SU_w(2) \times U_Y(1)$ formulation for quarks and leptons is shown.

1. INTRODUCTION

The rapid development of elementary particle physics in the last decade has presented us with a better defined panorama of this field. It is now accepted that the building blocks can be taken to be quarks and leptons and their interaction (gauge) fields. It is not surprising, then, that this is reflected in the titles of recent publications [for example, the books by Huang (1982), Close (1979), Field (1979), Okun (1982), and Halzen and Martin (1984)].

A family of elementary particle fields is now known to be composed of a left-handed neutrino, a right- or left-handed electron, and the (assumed left-handed) quarks. Recent analysis of strong interaction experiments makes it plausible that parity is not conserved in strong interactions, and in that case one would conclude that quarks could only be found in a well-defined helicity state, which, because of what is already known from weak interactions, will most probably be the left-handed state.

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In the case of quarks it has been found advantageous to assume exactly three colors [Greenberg (1964); see also Greenberg (1982)] and the additional rule that only "colorless" states can be seen experimentally.

The gauge interaction fields should fit the previous description, and a complete theory is expected to explain other properties, such as confinement, asymptotic freedom, and charges, and eventually, provided the dynamics is complete, should make it possible to compute masses, absolute decay ratios, etc.

The purpose of the present paper is to derive a unified theoretical framework which will also clarify the existence of families and provide a basis for grand unification.

Our starting point will be left-handedness considered as a basic fact and as such we will incorporate it in the wave equation itself, instead of selecting one particular solution to the standard Dirac equation. We will be able to do this for a massless neutrino; then the equation we will obtain will automatically ensure that neutrinos will be left-handed and massless. We obtain similar equations for left-handed fields to represent quarks and we will find that there are two ways of doing this, each of which can be realized in three different, but equivalent, forms, which we will then identify with the prototypes of the previously hypothetical u and d quarks in their three different colors. Quarks, being confined, therefore lose the absolute meaning of chirality. The use of the same procedure for the electron will nevertheless allow for right-handed electrons, too. We discuss the basic properties of these fields, their extension to other families of elementary particle fields, and their gauging properties, to arrive at the concept of charges (fractional and integer), weak charges, color charges, and a basic collection of interaction gauge fields, which will then provide a basis for both electroweak and grand unification.

At the end of this paper deal with the problem of asymptotic freedom and confinement within the present theory and a definition of elementary particle fields and of the observable (simple or composite) *free* elementary particles. The composite elementary particles, mesons and baryons, require the existence of correlation lengths of the same order of magnitude as the particle radii. The paper is divided into four main sections, which in fact could be presented independently, but since they support each other in a systematic manner, we have chosen to include them together.

The first part develops the mathematical framework needed to study the fields that can exist in physical space-time. Here, then, space-time is the basic notion for the descriptions of physical phenomena and not a secondary concept [which it could in fact be if a different philosophy of presentation were used, with matter fields as primary entities, as in the approach of Marlow (1982, 1984)]. In the next part we give a geometrical definition of the de Broglie phase, and study the consequences that follow if rotations in a reference space $R^{0,5}$ are quantized ($R^{0,5}$ corresponds to the use of complex algebra in the description of phenomena in the physical space-time $R^{1,3}$). In the part that follows, the Dirac equation is presented in multivector form in order to have a geometrical guide to generalizing it in (complex algebra) space-time. The generalization of the Dirac equation is made and the concept of symmetry-constrained dirac fields (diracons) is introduced. The remainder of the paper is devoted to the development of the theory of diracons, showing that they have the properties and symmetries we expect of leptons and quarks and their interaction fields.

2. VECTORS AND SPINORS IN COMPLEX SPACE-TIME AND THEIR SUBSPACES

2.1. Vectors

The multivectors are generated by the antisymmetric, Grassmann, outer product $\wedge$ of a basis set $\{\gamma_\mu\}$ in N dimensions

$$\gamma_{AB}=\gamma_A \wedge \gamma_B = \tfrac{1}{2}(\gamma_A\gamma_B - \bar{\gamma}_B\bar{\gamma}_A) \tag{1}$$

where

$$\gamma_A=\gamma_{\mu\nu\ldots\lambda} \qquad \text{and} \qquad \bar{\gamma}_A=(-\gamma_\mu)(-\gamma_\nu)\cdots(-\gamma_\lambda) \tag{1a}$$

Also $\tilde{\gamma}_A=\gamma_{\lambda\ldots\nu\mu}$.

The corresponding Clifford algebra is constructed using the Grassmann algebra and an inner (dot) product,

$$\gamma_A \cdot \gamma_B = \tfrac{1}{2}(\gamma_A\gamma_B + \bar{\gamma}_B\bar{\gamma}_A) \tag{2}$$

to define the total, or geometric, product:

$$\gamma_A\gamma_B=\gamma_A \cdot \gamma_B + \gamma_A \wedge \gamma_B \tag{3}$$

The metric of space-time $R^{1,3}$ ($\mu=0,1,2,3$) is defined through the inner product

$$g_{\mu\nu}=\gamma_\mu \cdot \gamma_\nu = \mathrm{diag}(1,-1,-1,-1) \tag{4}$$

If the multivector algebra C^N is considered as the complexifiction of $R^{m,n}(N=m+n)$, we require the concept of absolute value square $|\gamma_A|^2 = \gamma_A \cdot \gamma_A^\kappa$ (which is not restricted to positive values), where γ_A^κ is a multivector with all *coefficients* being the conjugate of those of γ_A. We can write formally for the complexification $\mathscr{D}_c \sim C^4$ of the space-time algebra $\mathscr{D}$

$$\mathscr{D}_c=\mathscr{D}+i\mathscr{D}=\mathscr{D}_R+\mathscr{D}_I; \qquad \mathscr{D}_R^*=\mathscr{D}_R, \qquad \mathscr{D}_I^*=-\mathscr{D}_I \tag{5}$$

$$\mathscr{D}=\mathscr{D}_+ + \mathscr{D}_-; \qquad \mathscr{D}_+=\bar{\mathscr{D}}_+, \qquad \mathscr{D}_-=-\bar{\mathscr{D}}_- \tag{6}$$

Also, $\mathscr{D}_- = \gamma_0 \mathscr{D}_+$. The even subalgebra $\mathscr{D}_+ \sim \mathscr{P}_c$ corresponds to the standard (three-dimensional) space. Also, if $\gamma_5 = \gamma_{0123}$, $\mathscr{D} = \mathscr{D}_1 + \gamma_5 \mathscr{D}_1$ is the main $U(1)$ operation of the algebra. $\mathscr{D}_1$ corresponds to the set {scalar + vectors + three-dimensional space planes}.

We can construct projection operators P_A with any of the multivectors γ_A: $\gamma_A^2 = 1$ (except $\gamma_A = 1$), which divide all elements into subsets (or components), which are the $A_{\text{even}}[P_A^+ = (1+\gamma_A)/2]$ and $A_{\text{odd}}[P_A^- = (1-\gamma_A)/2]$ parts:

$$P_A^+ P_A^- = P_A^- P_A^+ = 0, \qquad P_A^+ P_A^+ = P_A^+, \qquad P_A^- P_A^- = P_A^-, \qquad P_A^+ + P_A^- = 1 \tag{7}$$

For an algebra of dimension N the number p of independent P_A is p = integer part of $(N/2)$.

All multivectors are operators on themselves and on their spinors. The best-known examples are γ_0, generating the parity inversion **P**; γ_{123}, the time inversion **T**; γ_{0i}, the Lorentz boosts $\mathscr{L}$; γ_{ij}, the space rotations $\mathscr{R}$; γ_5, the duality transformation **D**; and $i\gamma_5$, the chirality projection.

The pseudoscalar unit is $\gamma_5 = \gamma_{\mu\nu\lambda\rho}\varepsilon^{\mu\nu\lambda\rho}/4!$ in space–time $R^{1,3}$, but it is simply i $(=\sqrt{-1})$ in $R^{0,5}$.

$\mathscr{D}_c$ can be regarded both as the complexification of the space–time multivector algebra or as a five-dimensional space whose even subalgebra corresponds to spacetime, as shown in Section 2.3.4.

Table I. Subspaces Nested As Even Subalgebras in the Complexification of Spacetime $\mathscr{D}_c$[a]

Dimensions		Spacetime multivectors						Smallest matrix representation	Spinor space
N	Symbol	S	V	BiV	TriV	TetraV	PentaV		
0	R	1						$R(1)$	Real numbers
1	C	1	$\mathrm{i}_1 = \gamma_{12}$					$C(1)$	Complex numbers
2	$\mathscr{P}$	1	$e_h = \gamma_{h3}$	$\gamma_{12} = e_{12}$				$R(2) = H_+$	R-Pauli spinors
3	$\mathscr{P}_c$	1	$\gamma_i^{\mathscr{P}_c} = \gamma_{01}$	γ_{ij}	$\gamma_5 = \gamma_{ijk}^{\mathscr{P}}$			$C(2) = H$	C-Pauli spinors
4	$\mathscr{D}$	1	$\gamma_\mu = \gamma_\mu$	$\gamma_{\mu\nu}$	$\gamma_{\mu\nu\gamma}$	$\gamma_5 = \gamma_{0123}$		$H^2 = C_+(4)$	Dirac spinors
5	$\mathscr{D}_c$	1	$\gamma_A = i_2\gamma_{\mu'}$	γ_{AB}	γ_{ABC}	γ_{ABCD}	$i_2 = i$	$C(4)$	Dirac spinors

[a] H^2 is the set of block diagonal and block antidiagonal matrices with nonzero blocks corresponding to the (complex) Pauli matrices H. The $R(n)$ are real $n \times n$ matrices, the $C(N)$ are complex $n \times n$ matrices. $\gamma_{\mu'} = \{\gamma_{123}, \gamma_0, \gamma_{01}, \gamma_{02}, \gamma_{03}\}$. Capital letter indices run from 1 to 5. Greek letter indices run from 0 to 3. Latin letter indices run from 1 to 3.

The subspaces $\mathscr{P}$ and C in Table I are not the restrictions of standard space $\mathscr{P}_c \sim R^{3,0}$ to two or one dimensions. They are the multivectors of, respectively, $\mathscr{P} \sim R^{0,2}$ and $C \sim R^{0,1}$.

2.2. Spinors

We define spinors as left (right) modules of multivectors; Keller et al. (1986) present spinors as the basic mathematical entities to construct a vectorial space, in an approach similar to Marlow's (1982, 1984) study of relativity from quantum theory. We will define the spinors of any complete set of 2^N multivectors, for example, $\mathscr{D}_c$, as a vectorial space $\mathscr{L}$ of dimension 2^p [p = integer part of $(N/2)$] with the closure property

$$\text{if} \quad \chi \in \mathscr{L} \quad \text{and} \quad M \in \mathscr{D}_c, \quad \text{then} \quad M\chi \in \mathscr{L} \tag{8}$$

and for the dual spinor space $\mathscr{L}^\dagger$

$$\text{if} \quad \chi^\dagger \in \mathscr{L}^\dagger \quad \text{and} \quad M \in \mathscr{D}_c, \quad \text{then} \quad \chi^\dagger M \in \mathscr{L}^\dagger \tag{9}$$

in such a way that we obtain $\mathscr{D}_c$ from the "outer" spinor product $\chi\chi^\dagger \in \mathscr{D}_c$ and the "inner" spinor product $\chi_\beta^\dagger \chi_\lambda = C_{\beta\lambda}$, with $C_{\beta\lambda}$ a complex number taken to be $\delta_{\alpha\beta}$ in Sections 2.3.1 and 2.3.2. If

$$M = M^{\alpha\beta}\chi_\alpha\chi_\beta^\dagger \tag{10a}$$

then closure is proven thus:

$$M\chi_\lambda = M^{\alpha\beta}\chi_\alpha\chi_\beta^\dagger\chi_\lambda = C_{\beta\lambda}M^{\alpha\beta}\chi_\alpha = \chi_\sigma \in \mathscr{L} \tag{10b}$$

and correspondingly for the $\chi_\alpha^\dagger \in \mathscr{L}^\dagger$. The $M^{\alpha\beta}$ are (a particular) matrix representation of the multivector Clifford algebra $M \in \mathscr{D}_c$.

2.3. Mathematical Properties

2.3.1. Classification of $\mathscr{L}_{\mathscr{D}_c}$ Spinors

For complex space-time the multivector $i\gamma_5$ plays a central role in the algebra; for this reason it is customary to define the main projectors Q_R and Q_L and name the two spinor subspaces generated by the Q on the spinor space $\mathscr{L}_{\mathscr{D}_c}$ left-handed L amd right-handed R, such that

$$\mathscr{L}_{\mathscr{D}_c} = \mathscr{L}_R + \mathscr{L}_L \tag{11}$$

For $\mathscr{D}_c$ = complex space-time with dimension $N = 5$, the number of basic spinors is 2^p, with p = integer part of $(N/2) = 2$; then we need *two* projection operators A, which will be either the chiral representation $A = (i\gamma_5$ and $i\gamma_{12})$ or the standard representation, where massive particles have already been defined, with $A = (i\gamma_{12}$ and $\gamma_0)$. The spinors will carry $n = 2^p = 4$

double indexes or, as customary, a single index α taking 2^p values ($\alpha = 1, \ldots, 2^p$).

The commonly employed projectors are the chirality projectors:

$$Q_{R/L} = (1 \pm i\gamma_5)/2 \tag{12}$$

the mass projectors

$$m_{+/-} = (1 \pm \gamma_0)/2 \tag{13}$$

and the spin projectors

$$S_{\uparrow/\downarrow} = (1 \pm i\gamma_1\gamma_2)/2 \tag{14}$$

The relations used to construct the starting multivectors will be

$$\gamma_0 i\gamma_5 \chi_L = -(\gamma_0 \chi_L) = -i\gamma_5(\gamma_0 \chi_L) \tag{15}$$

or

$$\gamma_0 i\gamma_5 \chi_R = +(\gamma_0 \chi_R) = -i\gamma_5(\gamma_0 \chi_R) \tag{16}$$

and

$$(i\gamma_5)(i\gamma_{12}\chi_L) = i\gamma_{12} i\gamma_5 \chi_L = -(i\gamma_{12}\chi_L) \tag{17}$$

or

$$(i\gamma_5)(i\gamma_{12}\chi_R) = i\gamma_{12} i\gamma_5 \chi_R = +(i\gamma_{12}\chi_R) \tag{18}$$

In the following, ch $= \{L, R\}$, $S = \{\uparrow, \downarrow\}$, and $m = \{+, -\}$.

2.3.2. *The Elements of $\mathscr{D}_c$ As a Linear Combination of Spinor Products $\chi\chi^\dagger \in \mathscr{D}_c$*

In the spin chiral representation we can construct $i\gamma_5$ as

$$i\gamma_5 \equiv \sum_S (\chi_{SR}\chi^\dagger_{SR} - \chi_{SL}\chi^\dagger_{SL}) e^{i\theta_5} \tag{19}$$

and

$$\begin{aligned} i\gamma_{12} &\equiv \sum_{\text{ch}} (\chi_{\uparrow\text{ch}}\chi^\dagger_{\uparrow\text{ch}} - \chi_{\downarrow\text{ch}}\chi^\dagger_{\downarrow\text{ch}}) e^{i\theta_{12}} \\ \gamma_0 &\equiv \sum_S (\chi_{SL}\chi^\dagger_{SR} + \chi_{SR}\chi^\dagger_{SL}) e^{i\theta_0} \end{aligned} \tag{20}$$

In this representation $\theta_{12} = \theta_0 = \theta_5 = 0$.

From the commutation and anticommutation relations with γ_5 and γ_{12} the vector basis set is found to be

$$\begin{aligned} \gamma_0 &= \chi_{\uparrow L}\chi^\dagger_{\uparrow R} + \chi_{\downarrow L}\chi^\dagger_{\downarrow R} + \chi_{\uparrow R}\chi^\dagger_{\uparrow L} + \chi_{\downarrow R}\chi^\dagger_{\downarrow L} \\ \gamma_1 &= -\chi_{\uparrow L}\chi^\dagger_{\downarrow R} - \chi_{\downarrow L}\chi^\dagger_{\uparrow R} + \chi_{\uparrow R}\chi^\dagger_{\downarrow L} + \chi_{\downarrow R}\chi^\dagger_{\uparrow L} \\ \gamma_2 &= i(-\chi_{\uparrow L}\chi^\dagger_{\downarrow R} + \chi_{\downarrow L}\chi^\dagger_{\uparrow R} + \chi_{\uparrow R}\chi^\dagger_{\downarrow L} - \chi_{\downarrow} R\chi^\dagger_{\uparrow L}) \\ \gamma_3 &= -\chi_{\uparrow L}\chi^\dagger_{\uparrow R} + \chi_{\downarrow L}\chi^\dagger_{\downarrow R} + \chi_{\uparrow R}\chi^\dagger_{\uparrow L} - \chi_{\downarrow R}\chi^\dagger_{\downarrow L} \end{aligned} \tag{21}$$

The scalar unit of the algebra should be written

$$1 = \sum_S \sum_{\text{ch}} \chi_{S,\text{ch}} \chi^{\dagger}_{S,\text{ch}}$$

2.3.3. Matrix Representations

If we define the column matrix of the basis $\chi_\alpha \in \mathscr{L}$ and the row matrix of the $\chi^{\dagger}_{\alpha} \in \mathscr{L}^{\dagger}$, the multivectors will be represented by the coefficients obtained from (21) and (10) as $\gamma_A \to M^{\alpha\beta}_{A}$, the usual matrices used in quantum mechanics.

The "standard" matrix representation of quantum mechanics corresponds to the choice of γ_0 and $i\gamma_{12}$ as projection operators and the corresponding spinors

$$\gamma_0 \chi_{S+} = \chi_{S+} \qquad \text{and} \qquad \gamma_0 \chi_{S-} = -\chi_{S-} \tag{22}$$

2.3.4. The Most Commonly Used Subspaces of C^4 and of $R^{1,3}$

First require the five basis vectors of $\mathscr{D}_c$ in the form $(R^{0,5})$

$$\gamma_v = \{i\gamma_{123}, i\gamma_0, i\gamma_{01}, i\gamma_{02}, i\gamma_{03}\} \tag{23a}$$

and the $\mathscr{D}_c$ multivector algebra

$$\mathscr{D}_c = \{1, \gamma_u, \gamma_{uv}, \gamma_{muv}, \gamma_{mnuv}, i\} \tag{23b}$$

in order that its even part corresponds to $\mathscr{D} = R^{1,3}$.

The standard reduction chain of even subalgebras

$$\mathscr{D}_c \to \mathscr{D} \to \mathscr{P}_c \to \mathscr{P} \to C^1 \to R^1$$

is straightforward (Table I); $\mathscr{D}_c$ and $\mathscr{D}$ share the same spinor basis. In our system $\mathscr{P}_c$ and $\mathscr{P}$ have two independent representations; they use either the spinor basis $(\chi_{\uparrow L}, \chi_{\downarrow L})$ or $(\chi_{\uparrow R}, \chi_{\downarrow R})$ (usually called dotted or undotted). The spinor basis of C^1 and R^1 is trivial.

It is important to emphasize here that all rotations in our representation of $\mathscr{D}_c$ are quantized in the standard formulation of quantum mechanics (Keller, 1985) and that changes in the spinor basis $\chi_\alpha \to \chi_\alpha + \varepsilon^{\beta}_{\alpha} \chi_\beta$ correspond to changes, using (10), in the multivector space. In general $\varepsilon^{\beta}_{\alpha} = \varepsilon^{\beta}_{\alpha}(\mathbf{x})$.

2.4. Covariant Vector and Spinor Derivatives

Following Hestenes (1966), define a differential operator $\square_i$ by

$$\square_i \phi = \partial_i \phi \tag{24}$$

where ϕ is a scalar and $\square_i$ maps scalars into scalars. For a vector field γ_j

$$\square_i \gamma_j = -L^k_{ij} \gamma_k \tag{25}$$

and for multivectors A and B

$$\Box_i(AB) = (\Box_i A)B + A\Box_i B \tag{26}$$

$$\Box_i(A+B) = \Box_i A + \Box_i B \tag{27}$$

In general, if $\mathbf{a} = a_j\gamma^j$, then

$$\Box_i \mathbf{a} = (\partial_i a_j + a_k L^k_{ij})\gamma^j \tag{28}$$

Hestenes (1966) uses this operator to discuss problems in general relativity [see also Hestenes and Sobczyk (1984) in this context].

For our spinor spaces $\mathscr{L}$ and $\mathscr{L}^+$ we can define

$$\Box_i\chi^\alpha = K^\alpha_{i\beta}\chi^\beta, \qquad \Box_i\chi^{\alpha+} = -K^\alpha_{i\beta}\chi^{\beta+} \tag{29}$$

where the $K^\rho_{i\delta}$ are related to the L^k_{ij} using the (representation-dependent) expansion (10) of the $\gamma^k = M^k_{\alpha\beta}\chi^\alpha\chi^{\beta\dagger}$.

3. ELEMENTARY PARTICLES AND THEIR INTERACTIONS

3.1. The Multivector Basis of the Dirac Form of the Theory of Elementary Particles

We have constructed in the preceding section a mathematical system for spinors and multivectors in physical space–time. We found it useful to enlarge it to a five-dimensional space $\mathscr{D}_c$ from a representation point of view: the basic spinor products $\chi_\alpha\chi^\dagger_\beta$ are combined, under the field of the complex numbers, to obtain the basic tetrad γ_μ; it was just natural to introduce the "complex" elements represented by $i\gamma_0$, $i\gamma_{123}$, $i\gamma_{01}$, $i\gamma_{02}$, and $i\gamma_{03}$ as a basis γ_A for $\mathscr{D}_c$. The pentad γ_A with metric

$$G_{AB} = \gamma_A\cdot\gamma_B = \mathrm{diag}(-1,-1,-1,-1,-1)$$

had as bivectors γ_{AB}:

$$\{\gamma_{AB}\colon \gamma^2_{AB} = -1\} = \{\gamma_5, \gamma_{12}, \gamma_{23}, \gamma_{31}\} \tag{30a}$$

and

$$\{\gamma_{AB}\colon \gamma^2_{AB} = 1\} = \{\gamma_j, \gamma_{0jk}\colon j \text{ and } k = 1, 2, 3\} \tag{30b}$$

The set (30a) generates rotations in $\mathscr{D}_c$, whereas the set (30b) generates hyperbolic transformations.

We can find a *physical* reason for using a complex space–time framework if we explore the consequences of the following postulate:

Postulate I. All rotations in $\mathscr{D}_c$ are quantized.

The quantization of the rotations $\theta_{ij} = 2\pi L_{ij}$ in the spacelike planes γ_{ij} corresponds to our standard quantization $L_{ij} = n\hbar$ of angular momentum L_{ij}. The rotation S in the (abstract in space-time) plane γ_5 corresponds to the de Broglie hypothesis if physical linear momentum $\mathbf{p}$ is mapped into a space-time three-vector

$$\mathbf{p}^D = p^\mu \gamma_\mu^D; \qquad \gamma_\mu^D \equiv \gamma_\mu \gamma_5 \tag{31}$$

which we will call the *geometrical momentum.* This mapping is a generalization of the Gibbs construction of polar $\mathbf{e}_i$ and axial vectors $\mathbf{e}_i \equiv \mathbf{e}_j \times \mathbf{e}_k$, but it differs in behavior because $\gamma_\mu^D = \gamma_\mu \gamma_5$, the axial vectors of relativity, do change sign under a reference frame inversion; moreover, γ_μ and γ_μ^D are both members of the odd multivector algebra of $\mathscr{D}$. The position $\mathbf{x}$ (standard) space-time vector $\mathbf{x} = x^\mu \gamma_\mu$ such that the outer product

$$\mathbf{x} \wedge \mathbf{p}^D = (\mathbf{x} \cdot \mathbf{p})\gamma_5 \tag{32}$$

is the (dual of the) action $\mathscr{S}$. The inner product is the (dual of the) angular momentum (in units of γ_5). The action is a scalar, which, according to our postulate, is quantized:

$$\mathscr{S} = \mathbf{x} \cdot \mathbf{P} + \phi(\mathbf{x}) = x^\mu p_\mu + \phi(\mathbf{x}) = nh$$

Here $\phi(\mathbf{x})$ is a phase angle corresponding to a gauge transformation.

In the following we will write the trivector linear momentum as $\mathbf{p}\gamma_5$ or $\mathbf{p}^D = \mathbf{pI}$.

The simple and compact form that our choice of the basis γ_A of $\mathscr{D}_c$ and the Postulate I allow for quantization is, besides the fact that $\mathscr{D} = \mathscr{D}_c^{\text{even}}$, our main justification for considering $\mathscr{D}_c$ as a five-dimensional space, isomorphous, but not equal from the geometrical point of view, to the complexification of the space-time multivector algebra $\mathscr{D}$.

3.1.1. The Multivector Dirac Equation

An observer in reference system $\mathscr{S}(\mathscr{S}')$ associates an energy-momentum vector $\mathbf{p}(\mathbf{p}')$ to an electron (in fact to any "elementary" particle of mass m_0)

$$p^\beta \gamma_\beta = p'^\alpha \gamma'_\alpha \tag{33a}$$

where the basis vectors of $\mathscr{S}'$ and $\mathscr{S}$ are related through a Lorentz transformation

$$\gamma'_\alpha = \mathscr{L}\gamma_\alpha \mathscr{L}^{-1}; \qquad \mathscr{L}\mathscr{L}^{-1} = \mathscr{L}^{-1}\mathscr{L} = 1 \tag{33b}$$

Observer $\mathscr{S}'$ is taken to be that where $\mathbf{P}' = m_0 c\gamma'_0$; then postmultiplying (33) by $\mathscr{L}$, we obtain an equation in multivector form relating the particle's system to the observer's system $\mathscr{S}$:

$$p^\beta \gamma_\beta \mathscr{L} = m_0 c \mathscr{L} \gamma_0 \tag{34a}$$

Introduce (see Hestenes, 1966) the Schrödinger operator $\hat{p}^\beta$,

$$\hat{p}^\beta \mathscr{L} \equiv \hbar \partial^\beta \mathscr{L} \mathsf{I} = p^\beta \mathscr{L}, \qquad \text{with } \mathsf{I}^2 = -1 \tag{34b}$$

to obtain the multivector Dirac equation

$$-\gamma_\beta \partial^\beta \mathscr{L} = m_0 c \mathscr{L} \gamma_0 \mathsf{I} \tag{34c}$$

where $\hbar = 1$ and I is some rotation plane. Hestenes (1966, 1975) proposes $\mathsf{I} = \gamma_{12}$, but for the analysis of the rest of this section we need I to commute with all bivectors according to our choice $\mathsf{I} = \gamma_5$ given by Postulate I. As a consequence, rest mass will not be a primitive concept, but the result of the interaction between left- and right-handed fields (see below).

The general solution $\mathscr{L} = LQ_0$ to the multivector equation (34), where $Q_0 = A \exp(-\mathsf{I}\mathbf{p} \cdot \mathbf{x}/\hbar)$, can be "gauged"

$$\mathscr{L} \to \Psi = A' \exp\{-\mathsf{I}[\mathbf{p} \cdot \mathbf{x} + \phi(\mathbf{x})]/h\} \tag{35}$$

if the differential operator is generalized to a covariant derivative $\gamma_\mu \partial^\mu \to \gamma_\mu D^\mu$. In (35) the more general gauge "angle" is

$$\phi(\mathbf{x}) = \phi_{\text{scalar}}(\mathbf{x}) + \gamma_5 \phi_{PS}(\chi) + \gamma_\mu \gamma_\nu \partial^\mu \Omega^\nu(\mathbf{x}) \tag{36}$$

The scalar part is usually interpreted as corresponding to the electromagnetic field; the pseudoscalar part we interpret (see below) as corresponding to the weak and color fields, and the bivector part to the gravitational field (see Keller 1984, 1985). That is, the interaction fields are given as boundary data to represent both the rest of the physical world and the physical effect of the particle on itself. As usual, the electromagnetic interaction appears as a (complex) phase factor, which will produce an "extra" energy-momentum $\partial^\mu e^{-\mathsf{I}\phi} \mathsf{I} = (eA_\mu/c)e^{-\mathsf{I}\phi}$; the A_μ are the components of the usual electromagnetic field vector. The weak and color fields produce an extra vector-"axial" vector energy-momentum and the gravitational field changes the local, fiducial, frame $\gamma_\mu \to \gamma''_\mu(x)$. The gravitational interaction arises because, in order to compensate such a gauge transformation, a vierbein is needed (Keller, 1984)

$$f_\mu = (f^0 e^{-\partial\Omega})_\mu = f^\alpha_\mu f^0_\alpha; \qquad g^0_{\alpha\beta} = f^0_\alpha \cdot f^0_\beta = \text{diag}(1, -1, -1, -1) \tag{37}$$

where the f^0 are locally Lorentzian tetrads,

$$g_{\mu\nu} = g^0_{\alpha\beta} f^\alpha_\mu f^\beta_\nu = [g e^{-2\Box\Omega}]_{\mu\nu} \tag{38}$$

defining a (gauge-invariant) gravitational "field"

$$\Phi^\nu_\mu = \partial_\mu \Omega^\nu + \partial^\nu \Omega_\mu - \delta^\nu_\mu \partial^\alpha \Omega_\alpha \tag{39}$$

which will obey, for self-consistency, the "field" equation

$$\Box^2\phi = 4G\pi(\mathbf{T} - \tfrac{1}{2}gT) \tag{40}$$

with $\mathbf{T}$ the energy-momentum stress tensor of the *total* sources. The origin of the color and of the electroweak interactions between different fields will be discussed below.

The standard form of the Dirac equation is obtained (see Casanova, 1976) using a spinor u (composite in our theory) such that $\gamma_0 u = u$ and $\mathbf{I}u = iu$, to define $\Psi u = \psi$ and $\Psi\gamma_0 \mathbf{I}u = -i\psi$; here ψ is now a particular spinor projected out of Ψ,

$$i\gamma_\mu\, \partial^\mu \psi = m_0 c\psi \qquad \text{or} \qquad D\psi = -im_0 c\psi, \qquad D^2 = \partial_\mu\, \partial^\mu \tag{41}$$

In our theory the choice $\mathbf{I} = \gamma_5$ in Postulate I is the only one allowing the choice (36) of the gauging. But the usual (column) spinors cannot simultaneously be eigenspinors of γ_0 and γ_5 (remember $\gamma_0\gamma_5 = -\gamma_5\gamma_0$); then in our formulation, for massive particles, we need a more general spinor, representing a collection of fields.

There are two main possibilities in our spinor system to represent the collection of spinor fields $\chi_\alpha^{(d)}$ of composite particles: (1) as a matrix u consisting of rows of $\chi_\alpha^{(d)}$ or (2) as a supercolumn Ψ of $\chi_\alpha^{(d)}$. Each of these possibilities has its own advantages. The practical use of u is that it can be operated on the right by the elements of γ_A (if u contains four columns or less in the 4×4 representation of $\mathscr{D}_c$). The use of the column Ψ is standard in most of the elementary particle literature.

In space-time the composite wave function u can be symbolically written $u = a^\alpha_{d_1}\chi^{d_1}\chi^+_\alpha$, where the linear combination of diracon fields $a^\alpha_{d_1}\chi^{d_1}$ is placed by χ^+_α $(\alpha = 1, \ldots, 4)$ in the αth column of u. This is a possible procedure to construct the massive electron wave equation in (34) corresponding to that used by Casanova (1976) to study the baryon and meson multiplets.

The standard procedure of constructing a supercolumn spinor will be used below (see also Keller, 1985) to give an explicit formulation of $SU(2)\times U(1)$ electroweak interactions, $SU(3)_{\text{color}}$ chromodynamics, and a unified presentation of $SU(3)\times SU(2)\times U(1)$ in terms of the gauging (36) of the diracon fields. In this theory $\mathbf{I} = \gamma_5$ in equation (34) will be used, corresponding to a plane of $\mathscr{D}_c$ in (23).

The mass term will appear (as usual) as an interaction between the right- and left-handed parts of the electron fields. Introducing first the Casanova (composite) spinor u: $\gamma_5 u = iu$, then the projection $\Psi u^d = \psi^d$ will be made and afterward the mass term will be introduced to recover the standard Dirac equation (because γ_0 and γ_5 cannot have simultaneous eigenspinors, since they do not commute).

3.2. A Generalization of the Dirac Equation

For a massless particle $D\psi_0 = 0$. The ψ_0 obeys the Klein–Gordon equation with general solution Φ:

$$\partial_\mu \partial^\mu \Phi = 0 \qquad [\text{or} \quad -\partial_\mu \partial^\mu \Phi = (m_0 c)^2 \Phi] \tag{42}$$

from which the Dirac solution is obtained using the Dirac operator $D_0 = \gamma_\mu \partial^\mu$ to project it out,

$$\psi_0 = D_0 \Phi \qquad [\text{or} \quad \psi_0 = (D_0 + m_0 ci)\Phi] \tag{43}$$

We will now use multivectors to generalize (41) and to develop the theory of symmetry-constrained Dirac particles (Keller, 1982a, b, 1984, 1985). For this purpose we generalize the Dirac construction to a differential operator D valued in the (complex) multivector algebra $\mathscr{D}_c$:

$$DD^* = D^* D = \partial^\mu \partial_\mu \tag{44}$$

The Klein–Gordon equation operator ($c = \hbar = 1$)

$$(\partial^\mu \partial_\mu + m^2) = (D^* + mi)(D - mi) \tag{45}$$

requires $-D^* m + mD = 0$; that is, either {"A": $D^* = D$ and $m \neq 0$ (Dirac's)} or {"B": any D obeying (44) if $m = 0$}.

Let us restrict ourselves to case B (massless particles) and a D where we change (one or) several of the vectors γ_μ into a more general element γ_A. A hint comes from the special role of $i\gamma_5$ in elementary particle physics and from the general solution Φ above. Then, if we define a set d of coefficients $\{t_\mu^d\}$ for the construction of a *diracon* operator D_d,

$$D_0 = \gamma_\mu \partial^\mu \to D_d = \left\{ \cos(n + t_\mu^d) \ \frac{\pi}{2} + i\gamma_5 \sin(n + t_\mu^d) \frac{\pi}{2} \right\} \gamma_\mu \partial^\mu \tag{46}$$

or

$$D_d = a_d(\mu) \gamma_\mu \partial^\mu = \partial_d^\mu \gamma_\mu; \qquad \partial_d^\mu \equiv a_d(\mu) \partial^\mu \tag{47}$$

With the choice of n and t_μ^d integers, we obtain a set of diracon massless fields with definite chiralty $i\gamma_5 \psi_d = \pm \psi_d$. In that case $a_d(\mu) = \pm 1$ or $\pm i\gamma_5$ provided we also restrict $t_\mu^d = 0$ or 1, in order not to mix different chiralties.

Each D_d is characterized by the family index n and the particle field type set $\{t_\mu^d\}$ occurring in $a_d(\mu)$. The solutions to the massless Klein–Gordon equation (42) projected for a particular diracon field (46) are explicitly given by the immediate integration of the symmetry-constrained Dirac equation (Keller, 1982, 1984),

$$D_d \psi_d = 0; \qquad D_d^\kappa = a_d^\kappa(\mu) \gamma_\mu \partial^\mu; \qquad \psi_d = D_d^\kappa \Phi \tag{48}$$

as

$$\psi_d^0 = B \exp(\mathbf{I} p_d^\mu x_\mu); \qquad p_d^\mu \equiv a_d(\mu) p^\mu \tag{49}$$

with $x_\mu = x^\nu g_{\nu\mu}$.

Before proceeding further, we must first allow for the gauging of (48) and (49). The wave function can be gauged by a phase angle $\phi_d(\mathbf{x})$ if the "free" particle operator (47) is extended to the covariant derivative

$$D_d = \left[\partial_d^\mu - \mathbf{I} \frac{e}{\hbar} A_d^\mu(\mathbf{x}) \right] \gamma_\mu \tag{50}$$

to obtain the gauged solutions of the generalized Dirac equation

$$\psi(\mathbf{x}) = B \exp\{\mathbf{I}(p_d^\mu x_\mu + \phi_d(\mathbf{x})]\} \tag{51}$$

with

$$\begin{aligned} A_d^\mu(\mathbf{x}) = {} & A_{d,\text{scalar}}^\mu(\mathbf{x}) + A_{d,\text{pseudoscalar}}^\mu(\mathbf{x}) i\gamma_5 \\ & + A_{\alpha\beta,\text{tensor}}^\mu(\mathbf{x}) \gamma^{\alpha\beta} \end{aligned} \tag{52}$$

and the multivector phase angles, generalizing the de Broglie phase,

$$\begin{aligned} \phi_d(\mathbf{x}) = {} & \phi_{d,\text{scalar}}(\mathbf{x}) + \phi_{d,\text{pseudoscalar}}(\mathbf{x}) i\gamma_5 \\ & + \phi_{d,\alpha\beta,\text{tensor}}(\mathbf{x}) \gamma^{\alpha\beta} \end{aligned} \tag{53}$$

Because both the coefficients $a_d(\mu)$ in (49) and the multivector ϕ_d commute with $\mathbf{I} = \gamma_5$, in the following we will replace $\mathbf{I}$ by its eigenvalues $\pm i$.

These solutions can be better arranged in families corresponding to a given value of n, with left-handed chirality (for $t_\mu^d = 0, 1$) and a corresponding set of antifamilies, with the negative quantum numbers n and t_μ^d, with right-handed chirality, as shown in Table II. Here a special collection of diracon fields has been made which will be useful (see below) for asigning a symmetry and name in accordance with the usual $SU_c(3) \times SU(2) \times U(1)$ standard theory classification.

The phase factors $\phi_d(\mathbf{x})$ in equation (51) will allow the "interaction" and a resulting "transformation" of each of the basic diracon fields (48) among themselves [according to a $U(1)$ scheme or grouped in sets with $SU(2)$ or $SU(3)$ schemes]; it will result that a full understanding of any one of the diracon fields and their identification with observed elementary particles can only be obtained if all particles are considered together. If we study each family by itself, in a first approximation, and consider the right- and left-handed electron fields together, we obtain a Spin(8) scheme similar to that discussed by Smith (1985), where it is shown, after some parametrization, to provide a sound description of observed elementary particle fields. In the rest of this paper we will develop from (46), (48), and (50) a basic physical scheme of the actual observable particles.

Table II. Allowed Sets of Symmetry-Constrained Quantum Numbers $\{t_\mu^d\}$ and $\{t_\mu'^d \equiv t_\mu^d + n\}$ for Chiral Fields Corresponding to the Electron Family ($n=-1$), Satisfying the Generalized Dirac Equation $D_d\psi_d = 0$[a]

t_0	t_1	t_2	t_3	t_0'	t_1'	t_2'	t_3'	Charge	Isospin	Color	Symbol	Name
0	0	0	0	−1	−1	−1	−1	−1	−1	—	e^-	Electron
2	1	2	2	1	0	1	1	+2/3	1	r	u_r	Up quark
2	2	1	2	1	1	0	1	+2/3	1	b	u_b	
2	2	2	1	1	1	1	0	+2/3	1	g	u_g	
0	0	1	1	−1	−1	0	0	−1/3	0	r	d_r	Down quark
0	1	0	1	−1	0	−1	0	−1/3	0	b	d_b	
0	1	1	0	−1	0	0	−1	−1/3	0	g	d_g	
2	1	1	1	1	0	0	0	0	0	—	ν_e	Neutrino

[a]The quantum numbers n, t_μ^d, and operator D_d are defined in equations (46)-(48) in the text. The negative value of n corresponds to the choice of e^- as reference. The charges are given by the average value $(t_1'+t_2'+t_3')/3t_0'$ as described by the explanation of (56) in the text. The isospin pairs are connected by a change in the t_μ' such that $|t_\mu^d - t_\mu^{d'}| = (2,1,1,1) \bmod 2$, and the color triplets by a change in the t_μ' such that $t_\mu^d - t_\mu^{d'} = t_\nu^{d'} - t_\nu^d$.

The physical origin of isospin is the set of diracon fields d with coefficients $a_d(\mu)$ in (48). The grouping of N diracon fields, $d = 1, \ldots, N$, in a subset gives rise to the $SU(N)$ symmetry with fundamental representation (Hermitian) matrices $\hat{T}_a$, as shown explicitly in the following sections. This grouping will allow the introduction of an isospin form of the gauge fields $A^\mu \to A^{\mu a}\hat{T}_a$.

4. CHIRAL GEOMETRY THEORY OF ISOSPIN. LAGRANGIAN FORMULATION OF THE THEORY OF DIRACONS

A formal presentation of the dynamics of symmetry-constrained Dirac particles or diracons can be given in terms of a Lagrangian for the collection of particles. We will deduce this Lagrangian from the equations of the preceding section and show that it corresponds to the postulated standard formulation of grand unified theories, as described, for example, in Close (1979), Field (1979), Okun (1982), and Halzen and Martin (1984).

Table II is useful for an overall presentation of the different particles but it does not show the main symmetries of each type of field in the clearest form. In order to do so, we will first discuss the space-time symmetries of the gauged fields shown in Table II generated by the quantum numbers t_μ^d (and n).

One should keep in mind that the Lorentz transformations $\mathscr{L}$ preserve the multivector character; in each of the different terms m-vectors are mapped by $\mathscr{L}$ into m-vectors, even if the "components" changes in the usual way; then the equations (48) are multivector form invariant. This can be used to explore their symmetries. The same is true under spatial rotations.

For the quarklike diracons a more symmetric formulation can be given if the spatial coordinates are transformed in such a way that the local direction of motion γ_v of the particle is defined to be $\gamma_v = (\gamma_1 + \gamma_2 + \gamma_3)/\sqrt{3}$ and the notation $\gamma_\mu^D \equiv i\gamma_5\gamma_\mu$ is used in such a way that we can explicitly exhibit the vector-(imaginary) axial vector momentum admixture and show that it is constant for each color of the diracon field.

Let us write in detail the energy momentum multivector p of every diracon field, including the different "colors" d red (r), blue (b), or green (g) of the quarks, according to formula (49) and Table II:

$$\begin{array}{ll}
\text{electron } e: & p_e = p^0\gamma_0 + p^v(\gamma_1+\gamma_2+\gamma_3)/\sqrt{3} \\
 & p_{\bar{\mu}}^{\mathrm{r}} = p^0\gamma_0 + p^v(\gamma_1^D+\gamma_2+\gamma_3)/\sqrt{3} \\
\text{quark } \bar{u}: & p_{\bar{\mu}}^{\mathrm{b}} = p^0\gamma_0 + p^v(\gamma_1+\gamma_2^D+\gamma_3)/\sqrt{3} \\
 & p_{\bar{\mu}}^{g} = p^0\gamma_0 + p^v(\gamma_1+\gamma_2+\gamma_3^D)/\sqrt{3} \\
 & p_{\bar{d}}^{\mathrm{r}} = p^0\gamma_0 + p^v(\gamma_1+\gamma_2^D+\gamma_3^D)/\sqrt{3} \\
\text{quark } d: & p_{\bar{d}}^{\mathrm{b}} = p^0\gamma_0 + p^v(\gamma_1^D+\gamma_2+\gamma_3^D)/\sqrt{3} \\
 & p_{\bar{d}}^{g} = p^0\gamma_0 + p^v(\gamma_1^D+\gamma_2^D+\gamma_3)/\sqrt{3} \\
\text{neutrino } \nu: & p_\nu = p^0\gamma_0 + p^v(\gamma_1^D+\gamma_2^D+\gamma_3^D)/\sqrt{3}
\end{array} \tag{54}$$

Here p^v is the three-momentum and p^0 is the energy. We can see that the energy-momentum vectors are all in different phases of the $p_\mu \to p_\mu^D$ rotations.

Let us now consider a gauge energy-momentum *vector* field $A^\mu\gamma_\mu$ added to the diracon fields with coupling constant Q_e; we find that, in order to modify the *vector* part of the momentum in an amount similar to that for an electron, with the energy-momentum components given in the same proportion to the time part and to the spacial parts (calling $\gamma_\perp$ a vector perpendicular to the direction of motion γ_v) for the electron

$$\mathbf{p}' = (p^0 + Q_eA^0)\gamma_0 + (p^v + Q_eA^v)\gamma_v + Q_eA^\perp\gamma_\perp$$

has components:

$$\begin{array}{ll}
\text{timelike} & \gamma_0 \cdot p' = p^0 + Q_eA^0 \\
\text{spacelike parallel} & \gamma_v \cdot \mathbf{p}' = p^v + Q_eA^v \\
\text{spacelike perpendicular} & \gamma_\perp \cdot \mathbf{p}' = Q_eA^\perp
\end{array} \tag{55}$$

all of them scalar quantities.

However, for a $\bar{u}$ quark (taking, for example, a red quark, the result being invariant with respect to color)

$$\gamma_v \cdot \gamma_v^{\bar{u}} = \frac{1}{\sqrt{3}}(\gamma_1 + \gamma_2 + \gamma_3) \cdot \frac{1}{\sqrt{3}}(\gamma_1^D + \gamma_2 + \gamma_3) = \frac{2}{3} + \frac{1}{3} i\gamma_5 \tag{56}$$

the scalar components will be affected by a factor of 2/3, for a down quark by a factor of 1/3, and for a neutrino by a factor 0.

Then if we make the obvious definition that only the *scalar* part of the gauge field, which can be treated on an equal basis for the electrons and for the quarks or the neutrino, is to be considered the gauge field A, the "charges" have to be Q_e, $\frac{2}{3}Q_e$, $\frac{1}{3}Q_e$, and 0, respectively. The pseudoscalar $i\gamma_5$ parts are to be treated on a different basis, and will be shown to correspond to the weak and color interactions.

In the full Lagrangian, to be introduced and discussed below, a first term equivalent to the standard Dirac matter-field Lagrangian

$$\mathscr{L}_m = i\bar{\psi} D_\mu \gamma^\mu \psi \tag{57}$$

is to be replaced by the corresponding expression for diracons:

$$\mathscr{L}_d = i\bar{\psi}\partial_\mu \gamma_d^\mu \psi \tag{58}$$

It is in this term of the Lagrangian where we have to introduce an electromagnetic [scalar part of ϕ in (36)] gauging with a coefficient e for the electron field, $2e/3$ for the (anti) up quark field, $e/3$ for the down quark field, and 0 for the neutrino field. Then in the gauge theory we are constructing, the charges for the $U(1)$ part of the gauge fields are the (*postulated usually*) integer, fractional, or zero values of the standard theory. In general our method will allow us to *develop* a gauge theory instead of postulating it, as in the standard approaches. In this form we are showing the physical origin of the various couplings of the gauge fields, and the role played by $i\gamma_5$ in it, as a part of the symmetry-constrained Dirac particle theory.

For this purpose the A field discussed above will have to be split into contributions, usually called B and W^3 in the literature, and new "charges" T^3 and Y are introduced with the standard notation

$$Q = T^3 + Y/2 \tag{59}$$

but the assignment of T^3 and Y to give our values of Q will be straightforward and its physical origin clear.

It is convenient to start with a rearrangement of a diracon field in groups which will show an explicit $SU(2) \times SU(3) \subset \text{spin}(8)$ symmetry as shown in Table II.

To start, we explore the $SU(2)$ relations; for each given family we can see that the addition of a set of symmetry coefficients $\{W^-\}=(0,-1,-1,-1)$, modulus -2, to the first row produces the last row and their addition to any one of the first group of three up quark fields produces one of the group of three down quark fields. That is: the same chiral phase change that takes the neutrino field into a left electron field will change an up quark into a down quark. The reverse process proceeds in the corresponding way. The "neutral" interaction will arise from a change in the phase of one of the partner fields canceling that of the change of the other.

In the language of bilinear spinor operators described before we could write all these processes in terms of spinors: if $\{\chi_\nu, \chi_u, \chi_d, \chi_e\}=\chi_a$ represent the neutrino, up quark, down quark, and electron fields, respectively, and their respective dual fields are $\{\chi^\dagger_\nu, \chi^\dagger_u, \chi^\dagger_d, \chi^\dagger_e\}=\chi^\dagger_a$, with the orthogonality condition $\chi^\dagger_a\chi_b=\delta_{ab}$, then the processes above can be described:

$$\hat{W}^-=w^-(\chi_e\chi^\dagger_\nu+\chi_d\chi^\dagger_\mu) \tag{60a}$$

$$\hat{W}^+=w^+(\chi_\nu\chi^\dagger_e+\chi_u\chi^\dagger_d) \tag{60b}$$

and the neutral interaction (to be combined with the electromagnetic)

$$\hat{W}^3=w^3\tfrac{1}{2}(\hat{W}^+\hat{W}^- - \hat{W}^-\hat{W}^+) \tag{60c}$$

provided that in all cases the spins of each spinor operator of the product are opposite, i.e., that the spin of the electron field created is opposite to that of the neutrino field annihilated, etc. These processes correspond to vector interactions with total spin one, equal to the change in spin of the field during the interaction.

What we will show below is the correspondence between the interaction fields and each product of an interaction operator, written here in a formal way. We should add at this stage that, besides spin, energy-momentum is being exchanged during the interaction, for example, a photon interacting with an electron with energy-momentum exchange q could be written

$$\hat{A}=\sum_p \hat{\chi}_{e(p+q,\mp s\pm 1)}\hat{\chi}^\dagger_{e(p,\mp s)} \tag{61}$$

stating that the electromagnetic interaction annihilates an electron of momentum p and spin component s and creates an electron of momentum $p+q$ and of opposite spin.

The color interaction will change one of the spacelike t_i^d indexes of the quarks from the value 1 to 0 and produce a value 1 for one of the other indexes (which was zero previously), or change the axial vector momentum of two of those indexes simultaneously to a total of the eight operations $\{1\to 2, 1\to 3, 2\to 3, \bar{2}\to\bar{1}, \bar{3}\to\bar{1}, \bar{3}\to\bar{2}, 11\to 22, 22\to 33\}$ corresponding to the

$SU(3)$ color symmetry; we can also write these results in a formal operator way if we add a color subindex to the quark fields; then

$$\hat{G}_{ab} = \hat{\chi}_{q,a}\hat{\chi}^{\dagger}_{q,b} \tag{62}$$

will correspond to a gluonic interaction changing color b into color a.

All these interactions in our diracon fields and in our chiral phase language correspond to a change in the wave function

$$\psi_d = u \exp(p^d \cdot x + \phi^0_d) = u \exp(\phi_d) \tag{63}$$

with u a spinor and the de Broglie phases ϕ_d the scalar → pseudoscalar parts of the products of the momenta [given by equations (58)] with the vector x. The de Broglie phases gauged by the ϕ^0_d also contain scalar and pseudoscalar parts. For the leptons the de Broglie phases are

$$\phi_{\text{electron}} = p^{\mu}x_{\mu} + \phi^0_e \tag{64}$$

$$\phi_{\text{neutrino}} = p^0 x_0 + i\gamma_5 p^k x_k + \phi^0_{\nu}, \qquad k = 1, 2, 3 \tag{65}$$

The spinor u for the electron can be left- or right-handed; for the neutrino, in order to satisfy equation (48) only the left-handed field is possible.

For the quarks, in order to preserve rotational symmetry, we need to show explicitly the gauge phase $\phi^0_{q,a}$ ensuring that the overall de Broglie phase is space-symmetric. This requires a complicated vector notation. If a space index is k (with values 1, 2, 3), a reference space index is $r = 1, 2, 3$ and a color index is a or b (with vâlues r, b, g), a set of three multivectors (vector + i axial vector, $i = \sqrt{-1}$),

$$e^a_k = c^{ar}_k \gamma_r; \qquad c^{ar}_k = \cos \omega_{rk}[\cos(\pi t^a_r/2) + i\gamma_5 \sin(\pi t^a_r/2)] \tag{66}$$

for each color a of a given quark, direction k in space, and quantum number t^a_r in Table II for reference space direction r, this reference space direction at an angle ω_{rk} with the observer's space coordinates k. This is a more general notation than that of equation (54), where, for simplicity, the particle was taken to move in a direction with all $\cos \omega_{rv} = 1/\sqrt{3}$. The c^{ar}_k are then the sum of a scalar and (i times) a pseudoscalar.

For the purpose of our formalism we need a duality-symmetric set of coefficients b^{ar}_k such that $c^{ar}_k + b^{ar}_k = \cos \omega_{rk}$, the ordinary cosine directors (no axial vector mixing).

In terms of the multivectors (66) the de Broglie phases for the quarks are

$$\text{up quark} \qquad \phi_{u,a} = p^0 x_0 + c^{ar}_k p^k x_r + b^{ar}_k \phi^k x_r + \phi^0_{u,a} \tag{67}$$

$$\text{down quark} \qquad \phi_{d,b} = p^0 x_0 + c^{br}_k p^k x_r + b^{br}_k \phi^k x_r + \phi^0_{d,b} \tag{68}$$

The constants c^{ar}_k are different for up quarks and for down quarks, corresponding to the t^a_d quantum numbers.

Now, the phase angles ϕ_d^0 can either change the scalar-pseudoscalar structure of the de Broglie phases or leave them with the same structure. In the first case we have a change of the particle's nature (the resulting wave function will obey a different wave equation), and in the second case we have a type-conserving interaction. For this purpose we construct a Lagrangian which is invariant to the changes of the phase structure of the different $\phi_d = p_d^{\mu} x_{\mu} + \phi_d^0$ shown above. We do this here using matrix notation for isospin to conform to the usual expressions of the standard theory.

We have two options: either we put all eight (left-handed) fields together in a column isospin matrix and show the pair of fields connected by $SU(2)$ symmetries and the triad of fields corresponding to $SU(3)_{\text{color}}$ interactions, or construct directly the $SU(2)$ doublets and the $SU(3)$ triplets. The first possibility is the more physical one (Keller, 1985), although it requires a less familiar and more elaborate notation. The second one shows the symmetries in their clearest form and conforms to most current papers; for this reason we will use it here, at the expense of writing more terms in the Lagrangian.

First we need to state clearly that for each term ϕ^B in the phase angles in (63) we need to add a term $-gB$ in the (covariant) derivative, as usual in gauge theory, and a kinetic energy term $\frac{1}{4} F_B^{\mu\nu} F_{\mu\nu}^B$ for the gauge field B, again in the usual way. But what is new in our approach is that if the gauge angles in (63) change the scalar-pseudoscalar structure of ψ_d, then the d field has been transformed into a new field, say d'; then in the Lagrangian the covariant derivative term will acquire an index $a = 1, \ldots, N^2 - 1$, indicating that it corresponds to an $SU(N)$ type-changing interaction field and it will appear in the covariant derivative and in the Lagrangian multiplied by an $SU(N)$ matrix T_a; then $-gB \to -gB^a T_a$. The representation of the T_a matrices required here is the isospin (or color) step-up or step-down form.

Let us illustrate this for the electron-neutrino left-handed pair. We start with the definition of the $SU(2)$ isospin pair and its kinetic energy Lagrangian density

$$\hat{K} = \mathbf{1}\gamma_\mu \partial^\mu; \qquad L = \begin{pmatrix} \psi_{\nu_L} \\ \psi_{e_L} \end{pmatrix}; \qquad \mathscr{L}_{K,L} = i\bar{L}\mathbf{1}\gamma_\mu \partial_\mu L \tag{69}$$

and, in order to make it gauge invariant, we transform the kinetic energy operator into the standard covariant derivative

$$\begin{aligned} \hat{K}^{st}_{w,y} &= \begin{pmatrix} \gamma_\mu(i\partial^\mu - gW_3^\mu - \frac{1}{2}g'B^\mu) & g\gamma_\mu W_+^\mu \\ g\gamma_\mu W_-^\mu & \gamma_\mu(i\partial^\mu + gW_3^\mu - \frac{1}{2}g'B^\mu) \end{pmatrix} \\ &= \gamma_\mu(i\mathbf{1}\partial^\mu - gW_a^\mu T^a - g'YB^\mu) \end{aligned} \tag{70}$$

with $T^a = \frac{1}{2}\tau^a$, the τ^a being the Pauli matrices, and Y a charge matrix; to obtain the $SU(2)$ gauge-invariant Lagrangian density

$$\mathscr{L}_{K,L} = \bar{L} K_w L \tag{71}$$

The Lagrangian density (71) will have three contributions, $\bar{e}_L e_L - \bar{\nu}_L \nu_L$, $\bar{\nu}_L (W_+) e_L$, and $\bar{e}_L (W_-) \nu_L$, which have some special properties: the first term is a concerted scattering where the energy-momentum given to one of the leptons is withdrawn from the other; the second asserts that $(W_+) e_L$ behaves like a neutrino, and the third that $(W_-) \nu_L$ behaves like a (left-handed part of an) electron. In the diracon theory we can immediately keep the definition of L, but in principle it was not needed to state that the neutrino was left-handed, because it *must* be left-handed to obey its wave equation. The kinetic energy operator is slightly changed through the use of the substitution $\gamma_\mu \to \gamma_\mu^d = a^d(\mu)\gamma_\mu$ defined in (46) to read for free fields

$$\hat{K} = \begin{pmatrix} i\gamma_\mu^{(\nu)} \partial^\mu & 0 \\ 0 & i\gamma_\mu^{(e)} \partial^\mu \end{pmatrix} \tag{72}$$

before it is explicitly made gauge invariant, and the $SU(2)$ part in the diracon theory

$$\hat{K}_w = \begin{pmatrix} \gamma_\mu^{(\nu)}(i\partial^\mu - gW_3^\mu) & -\gamma_\mu [a^{(\nu)}(\mu) p_f^\mu - a^{(e)}(\mu) p_i^\mu] \\ \gamma_\mu (a^{(e)}(\mu) p_f^\mu - a^{(\nu)}(\mu) p_i^\mu) & \gamma_\mu^{(e)}(i\partial^\mu + gW_3^\mu) \end{pmatrix} \tag{73}$$

Here again, in the notation of our formalism, we have the following concerted interactions: the "neutral" interaction where the moment given to the electron field $(-gW_3^\mu \gamma_\mu)$ cancels that given to the neutrino field; the "positively charged" interaction $[a^{(\nu)}(\mu) p_f^\mu - a^{(e)}(\mu) p_i^\mu]$ where an electron of initial moment $p_i^\mu \gamma_\mu$ appears in the final state as a neutrino field momentum $a^{(\nu)}(\mu) p_f^\mu \gamma_\mu$; and, finally, the reciprocal, "negatively" charged interaction where the initial state is a neutrino and the final state is an electron. All this is through changes in the vector–axial vector coefficients $a^d(\mu)\gamma_\mu$. The equivalence of (73) and (70) is immediate if we now apply both to the L wave function, which is an eigenfunction of $i\gamma_5$ [or equivalently of the $a^d(\mu)$]: $-a^d(\mu) L = L$.

We finally obtain, as expected, the equivalence

$$gW_+^\mu = (p_f^\mu - p_i^\mu) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad gW_-^\mu = (p_f^\mu - p_i^\mu) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{74}$$

A similar procedure transforms a set of three colors of a quark field among themselves. Again it is advantageous to write the representation of $SU(3)_{\text{color}}$ in the eight (step up, step down, and color neutral interactions) between the three pairs of colors r–b, r–g, and b–g.

Our procedure has been the following: (1) write the phase angles of the de Broglie phases [equation (53)], (2) introduce a covariant derivative for each component of the gauge phase angles [equation (52)], and (3) write the concerted pairs (or trios) of particle fields in the form of isospin or color multiplets, with the corresponding electroweak and color charges.

The complete Lagrangian density is

$$\mathscr{L} = \mathscr{L}_l + \mathscr{L}_B + \mathscr{L}_W + \mathscr{L}_Q + \mathscr{L}_{\text{mass}} \tag{75}$$

with

$$\mathscr{L}_l = \bar{L}\hat{K}_W L + \bar{\psi}_{e_R}\gamma^\mu[i\partial_\mu - g'B_\mu]\psi_{e_R} + \bar{L}\gamma^\mu(-\tfrac{1}{2}g'B_\mu)L \tag{76}$$

$$\mathscr{L}_B = -\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{77}$$

$$\mathscr{L}_W = -\tfrac{1}{4}\mathbf{W}_{\mu\nu}\cdot\mathbf{W}^{\mu\nu} \tag{78}$$

$$\mathscr{L}_{\text{mass}} = -G_e(\bar{L}\phi_h\psi_{e_R} - \bar{\psi}_{e_R}\phi_h^\dagger L) \tag{79}$$

$$+|(i\partial_\mu - g\mathbf{T}\cdot\mathbf{W} - g'YB_\mu/2)\phi|^2 - V(\phi)$$

$$\mathscr{L}_q = \bar{q}\hat{K}_{W,Y,G}\,q - \tfrac{1}{4}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}^{\mu\nu} \tag{80}$$

where we have introduced the kinetic energies of the gauge fields, the Higgs fields discussed below, and the short-hand $\hat{K}_{W,Y,G}$ for the kinetic energy term of the quarks with weak, $U(1)$, and color interactions:

$$\hat{K}_{W,Y,G} = \gamma_\mu \begin{pmatrix} P^\mu_{ur} & P^\mu_{urb} & P^\mu_{urg} & P^\mu_{udr} & — & — \\ P^\mu_{ubr} & P^\mu_{ub} & P^\mu_{ubg} & — & P^\mu_{udb} & — \\ P^\mu_{ugr} & P^\mu_{ug} & P^\mu_{ug} & — & — & P^\mu_{udg} \\ P^\mu_{dur} & — & — & P^\mu_{dr} & P^\mu_{drb} & P^\mu_{drg} \\ — & P^\mu_{dub} & — & P^\mu_{dbr} & P^\mu_{db} & P^\mu_{dbg} \\ — & — & P^\mu_{dug} & P^\mu_{dgr} & P^\mu_{dgb} & P^\mu_{dg} \end{pmatrix} = \gamma_\mu \mathbf{P}^\mu_q \tag{81}$$

Here we have used, for the gauged momenta of color a quark q,

$$P^\mu_{qa} = ia_{qa}(\mu)\partial^\mu - gT^3_q W^\mu_3 - \tfrac{1}{2}g'Y_qB^\mu - g_G G^\mu_{aa} \tag{82}$$

with the definition

$$Q_q = T^3_q + Y_q/2 \tag{83}$$

For the color interaction between like quarks of colors a and b with initial momenta p_i and final momenta p_f,

$$P^\mu_{qab} = a_{qa}(\mu)p^\mu_f - a_{qb}(\mu)p^\mu_i \tag{84}$$

And for the weak interaction between quarks of types q and q' corresponding to the same color a

$$p^{\mu}_{qq'a} = a_{qa}(\mu)p^{\mu}_{f} - a_{q'a}(\mu)p^{\mu}_{i} \tag{85}$$

Other interactions, for example, simultaneous color and type change, are not included here, for simplicity, but this kind of q–q' scattering can occur as a higher order process and represented here as a set of two more entries in the matrix (81).

Again, as in the case of leptons, the identification of the standard W^{μ}_{+}, W^{μ}_{-}, and G^{μ}_{ab} fields can be done once the result of operating with $a_{qa}(\mu)$ on the wave function ψ_{qa} is known.

The need of a colorless combination c of quarks in order to make $\mathscr{L}_{\text{quark}}$ rotation invariant imposes a bound state between quarks adding up to $c = a\bar{a}$ or $c = r + b + g$, that is, either

$$a_{qa}(\mu)p^{\mu}_{q} + a_{\bar{q}\bar{a}}(\mu)p^{\mu}_{q} = \cos\phi^{\mu}p^{\mu} \tag{86}$$

for a meson state, or

$$a_{q_1 r}(\mu)p^{\mu}_{1} + q_{q_2 b}(\mu)p^{\mu}_{2} + a_{q_3 q}(\mu)p^{\mu_3} = \cos\phi^{\mu}p^{\mu} \tag{87}$$

for a baryon, showing that the momenta of the component quarks are not independent at any time. The hadron's momenta include the gluon momenta, which in turn, as shown in (84), depend on the quark momenta, the situation being very complex because the gluon–gluon interaction is to be included, as discussed in quantum chromodynamics. The intensity of the gluon fields are then fixed by the requirements of colorless elementary composite particles, the hadrons, and because this intensity is given by the gauge field equations relating it to the sources; this in turn generates a distance parameter, the size of the hadron or equivalently the range of the gluon field, which ensures that the hadron can be considered colorless (Keller, 1984).

There are then two types of elementary particles: the quanta of the lepton fields and the composite elementary particles, the hadrons, which are composite but cannot be divided without rotational symmetry being violated.

Equations (84) and (85) show explicitly the role of chiral symmetry in generating color, charge, and weak charge.

To understand the structure of the $\mathscr{L}_{\text{mass}}$ Lagrangian, we must recall that the γ_{μ} anticommute with $i\gamma_5$; for this reason the gradient operator changes a right (left)-handed field into a left (right)-handed field.

Then we have as the only choice for the neutrino left-handed field

$$\partial^{\mu}_{(\nu)}\gamma_{\mu}\psi_{\nu_L} = 0 \tag{88}$$

but for the electron field we have the more general possibility of relating the left- and right-handed fields

$$\gamma_\mu \, \partial^\mu \psi_{e_L} = me^{i\theta}\psi_{e_R} \qquad \text{and} \qquad \gamma_\mu \, \partial^\mu \psi_{e_R} = me^{i\theta}\psi_{e_R} \tag{89}$$

Then the possibility of the existence of both (free) left- and right-handed electron fields allows the introduction of a new (mass) parameter, thus breaking the $SU(2)$ symmetry between the electron and the neutrino.

The expectancy value of γ_0 is the overlap of left-handed and right-handed components, so $\psi^+\gamma_0\psi$ is proportional to the mass which the field can acquire. A common normalization is $\bar{\psi}\psi = 2m$. This is clearly seen in the Weyl representation of the γ_μ.

We have here a new type of gauge freedom where a combination of left- and right-handed fields can be mapped into itself. In the case of bound particles (always the case of quarks) the distinction between left-handed and right-handed fields vanishes because of the presence of the interaction fields in the momentum operator; besides, the kinetic energy of the particle and of the gauge field (gluons, etc.) will have to be added to the center-of-mass (rest) energy of the composite particle (proton, meson, etc.).

The Higgs mechanism has to be chosen to express (88) and (89) in (79) because the left- and right-handed fields are independent and for the obvious reason that it also explains the masses of the W within the Glashow–Weinberg–Salam theory.

In order to proceed with the discussion of the correspondence with the standard $U(1)\times SU(2)\times SU(3)$ color–electroweak interaction (Greenberg, 1982; Fritzch and Minkowski, 1984; Georgi and Glashow, 1974; Georgi, 1975; Salam, 1968; Weinberg, 1967), we need to identify the scalar Higgs field. This is easier if we first write it in a formal, spin operator way. All the interactions above were required to simultaneously change the spin of the interacting particles, but we can also construct new interaction operators with the (opposite) requirement that the spin is conserved during the interaction,

$$\hat{H} = h^{ff'}\hat{\chi}_{f,s}\hat{\chi}^\dagger_{f',s} \tag{90}$$

Here the operator $\hat{H}$ will change the initial field into the final field with the same spin, but f and f' need not be the same. Then the operator $\hat{H}$ will carry a new isospin I, equal to $I_{f'}+I_f$, in the same way as the W operators above carried isospin or the G operators carried color. A neutrino-electron h, for example $\hat{H}_{e\bar{\nu}}$ and $\hat{H}_{\nu\bar{e}}$, will carry one unit of isospin; four of those scalar operators can be constructed corresponding to the four pairs $e\bar{\nu}$, $\nu\bar{e}$, $e\bar{e}$, $\nu\bar{\nu}$. The last two have zero isospin, whereas the previous ones have -1 or $+1$ isospin, respectively. This is the origin, within chiral geometry theory, of the isospin of the Higgs fields. Their expression in terms of our diracon field and their chiral phases are given by (90).

This scalar field will present an asymmetry with respect to the chiral set [spin(8)] of left-handed lepton and quark fields, because the electron field can be both left- and right-handed (e_L and e_R) and two terms will contribute in this case. Then the uncharged scalar, zero isospin, field will break the isospin symmetry among the scalar fields, due to the interaction between e_L and e_R.

In the matrix notation above $\chi_d^\dagger = (\ldots, \chi_d^\dagger, \ldots)$ is orthogonal by construction to $\chi_{d'}$ if $d \neq d'$. But the existence of nondiagonal terms in the Lagrangian shows that they are not physically independent; their relationships are explicitly formulated in the theory.

The Lagrangian (75) should be extended to include antiparticles. The particle–antiparticle formulation could look somewhat different for the electron field and for the neutrino or quark fields. We obtain the electron wave function from the even sum $e_L + e_R$ and the positron from the odd sum $e_L - e_R$; we see that there is a difference of $\pi/2$ in the relative chiral phases determining the character of the fields.

For the neutrino the phase difference is the same, but is usually given explicitly in the wave equation; for example, the positive energy solution of the neutrino, $E = |\mathbf{p}|$, satisfies

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}\chi = -\chi \tag{91}$$

corresponding, then, to the left-handed field, and the antineutrino, $E = -|\mathbf{p}|$, satisfies

$$\boldsymbol{\sigma} \cdot (-\hat{\mathbf{p}})\chi = \chi \tag{92}$$

and corresponds to a right-handed field. In both cases, for the particle–antiparticle, the change in phase is $\pi/2$ in the chiral plane $\psi_R \wedge \psi_L$, and there is no basic difference.

We end this section with two remarks about the geometrical interpretation which has been generated.

We first recall the notion of parity inversion $\mathbf{P}$ or space conjugation, consisting in the reversal of all the *space*like vectors of a multivector A; from the anticommutativity of the γ_μ

$$\mathbf{P}\gamma_0 \to \gamma_0, \qquad \mathbf{P}\gamma_k \to -\gamma_k = \gamma_0 \gamma_k \gamma_0 \tag{93}$$

Then, in general

$$\mathbf{P}A = \gamma_0 A \gamma_0 = A^P \tag{94}$$

and the notion of Hermitian conjugation, from (1a) and (94) as product reversal plus parity inversion, gives

$$A^\dagger \equiv \gamma_0 \tilde{A} \gamma_0 \tag{95}$$

[in the algebra defined in (23) parity inversion will change i into $-i$].

It is very important to understand the geometrical origin of the currents appearing in the Lagrangians (75). The normalization $\bar{\psi}\psi = \psi^{\dagger}\gamma_0\psi = 2m$ introduced above is the magnitude of a vector $J_0' = \Psi\gamma_0\Psi$ according to (34). For the system where the particle is at rest it is just two times the rest mass m of the particle and, because the Lorentz transformations are isometries of the vector algebra, it is the value in any (observer's) system S. The components of that vector in S are $J_0' \cdot \gamma_\mu$ or

$$\begin{aligned} J_0' \cdot \gamma_\mu &= -e(\tilde{\Psi}\gamma_0\Psi\gamma_\mu)_s = -e(\gamma_0\Psi^{\dagger}\gamma_0\gamma_0\Psi\gamma_\mu)_s \\ &= -e(\Psi^{\dagger}\gamma_0\gamma_\mu\Psi)_s \end{aligned} \tag{96}$$

where the subscript s stands for scalar part, and (95) has been used (see Hestenes, 1966, p. 44); premultiplying (96) by the unit dual (row) spinor $u^{\dagger}$ and postmultiplying it by u, we obtain the usual definition of the (conserved) current

$$j_\mu = -e\bar{\psi}\gamma_\mu\psi \tag{97}$$

used in the Lagrangians.

We can obtain a deeper geometrical insight if we analyze the example of the plane wave solutions F of the field intensities of the free Maxwell equations

$$\Box F = 0; \qquad F = fe^{\gamma_5 k\cdot x} \tag{98}$$

the bivector f and the wave vector k being constant and obeying the equation

$$\mathbf{k}f = k_0 f; \qquad k\gamma_0 = k_0 + \mathbf{k} \tag{99}$$

k_0 being a scalar and $\mathbf{k}$ a space vector with the conditions $k_0 = \pm|\mathbf{k}|$, obtained by multiplying (99) by $k_0 + \mathbf{k}$. If we apply to (99) the partity operation, f is transformed into f^P obeying

$$k_0 f^P = -\mathbf{k}f^P \tag{100}$$

showing that f and f^P behave as photon fields with the same energy and opposite momenta. On writing $f = \mathbf{e} + \gamma_5\mathbf{b}$, where $\mathbf{e}$ and $\mathbf{b}$ are space vectors, we have that equation (99) corresponds to $k_0\mathbf{e} = \gamma_5\mathbf{kb}$ and $k_0\mathbf{b} = -\gamma_5\mathbf{ke}$, showing that $\mathbf{k}$, $\mathbf{e}$, and $\mathbf{b}$ are mutually perpendicular, $\mathbf{e}^2 = \mathbf{b}^2$, and $\mathbf{e} \cdot \mathbf{b} = 0$. The factor $e^{\gamma_5 k\cdot x}$ in (98) shows that $\mathbf{e}$ and $\mathbf{b}$ are rotating into each other with a phase angle $\phi = k \cdot x$ and that in the free field f^P the rotation takes place in the opposite sense $\phi^P = -\phi = -k \cdot x$, corresponding to the two possibilities of circularly polarized light. For the photon field $p = \hbar k$ is the linear momentum; then we find again that the de Broglie phase $\phi = p \cdot x/\hbar$ corresponds to a duality rotation. The electromagnetic field, as in fact all gauge fields in our theory, adjusts the phases of the duality rotations between accelerated charged particles when photons are emitted or absorbed or when a particle is to be described in reference to other charged particles.

5. SUMMARY

We have given a self-consistent presentation of the motivation, formalism, and development of the theory of symmetry-constrained Dirac particles (diracons). Several approaches have been combined in the theory: the mathematical framework of the Clifford–Grassmann algebra of spinors and multivectors, the geometrical interpretation of the de Broglie phase $p \cdot x/\hbar$, the multivector formulation of the Dirac equation, the generalization of the Dirac "square root" procedure valued in the space–time complex multivector algebra where the vector–axial vector duality is explicitly included, and, finally, the gauging of individual fields and the collections of diracon fields to show the chiral origin of isospin and color. The Higgs field, carrying isospin in the sense here introduced, appears as a spin-conserving interaction connecting pairs of diracon fields; in particular, left-handed (e_L) and right-handed (e_R) fields produce a composite field ($e^- = e_R + e_L$ or $e^+ = e_R - e_L$) where it is possible to "cancel" the spacelike momentum and obtain a particle at rest with rest mass m. The Higgs fields are therefore the result of the possibility of formulating a Lagrangian term which is invariant to rotations in the abstract left chirality-right chirality plane (these rotations are generated by $i\gamma_{123}$ in the multivector Clifford algebra of space–time).

From the three types of fields we have discussed so far only the first and the third could be observed as free fields: massless neutrinos or massive electrons. The second type, the quarks, will break rotational invariance. But if we demand that we will only work with a *composite* field where a combination of particles of the quark type is made such that the group is no longer rotational symmetry-breaking, then this second type of field may also be observed as a free composite field.

We arrive then at a new concept. Ordinary composite particles, such as atoms and nuclei, can be split, when energy is available, into smaller components, whereas the composite elementary particles cannot, even if enough energy were available, unless a quark and antiquark are simultaneously created to restore rotational symmetry. This new type of particle, which, of course, corresponds to hadrons, will require, in order to preserve Lorentz and rotational symmetry, that three quarks (or a quark–antiquark pair) be together, as a minimum, in a small volume of space where there should be some coherence among the three quarks. This gives rise to a new type of interaction where each quark is constantly related to the other two in such a way that no particular "color" can be singled out. To achieve this we need to associate each quark to a number of quanta of a symmetry-constrained gauge field with the complementary colors, gluons, all together adding up to the hadron's mass.

We remark in concluding that the theory of diracons offers enough possibilities to provide a basis for the study of elementary particles and their interaction fields within the frame of physical space-time.

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